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# THE CONFORMAL RADIUS AS A FUNCTION AND ITS GRADIENT IMAGE

BY

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ABSTRACT

Let $\Omega$ be a domain in $\overline{\mathbb{C}}$ with three or more boundary points in $\overline{\mathbb{C}}$ and $R(w,\Omega)$ the conformal, resp. hyperbolic radius of $\Omega$ at the point $w \in \Omega \setminus \{\infty\}$. We give a unified proof and some generalizations of a number of known theorems that are concerned with the geometry of the surface $S_\Omega = \{(w,h) \mid w \in \Omega, h = R(w,\Omega)\}$ in the case that the Jacobian of $\nabla R(w,\Omega)$, the gradient of $R$, is nonnegative on $\Omega$. We discuss the function $\nabla R(w,\Omega)$ in some detail, since it plays a central role in our considerations. In particular, we prove that $\nabla R(w,\Omega)$ is a diffeomorphism of $\Omega$ for four different types of domains.

## 1. Introduction

Let $D$ denote the open unit disc and $\Omega$ a domain in $\overline{\mathbb{C}}$ with three or more boundary points in $\overline{\mathbb{C}}$. According to the Riemann mapping theorem, if $\Omega$ is

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simply connected and $w \in \Omega \setminus \{\infty\}$ is fixed, then there exists a unique conformal map $f(w,\cdot)$ of $D$ onto $\Omega$ such that

$$f(w,0) = w \quad \text{and} \quad \left.\frac{df(w,z)}{dz}\right|_{z=0} = f_z(w,0) > 0.$$

This quantity $f_z(w,0)$ is called the conformal radius of $\Omega$ at the point $w$ and will be denoted in the following by $R(w,\Omega)$. If $\Omega$ is as above but not simply connected, the generalization of Riemann's mapping theorem due to Poincaré (see, e.g., [1] and [11], p. 255) asserts that there exists a unique universal covering map of $D$ onto $\Omega$ which has the same normalization as the above conformal map. In this case the quantity $f_z(w,0)$ is called the hyperbolic radius (see, e.g., [6]). We will use the abbreviation $R(w,\Omega)$ likewise and use the terminus hyperbolic radius in any case.

Note that $R(w,\Omega) = 1/\lambda_\Omega(w)$, where $\lambda_\Omega$ is the density of the Poincaré metric with the curvature $K = -4$, i.e. the hyperbolic radius $R$ satisfies Liouville's equation

$$R(w)\Delta R(w) = |\nabla R(w)|^2 - 4, \quad w \in \Omega,$$

where $\Delta R$ denotes the Laplacian of $R$ and $\nabla R$ its gradient.

Concerning the behaviour of the hyperbolic radius near the point at infinity, it may be interesting that in a neighbourhood of infinity the asymptotic relation

$$R(w,\Omega) = \frac{|w|^2}{C(\mathbb{C}\setminus\Omega)} + O(1)$$

is valid, where $C(A)$ denotes the capacity of the plane set $A$ (see [11], p. 313). To avoid unnecessary complications we, from now on, do not explicitly mention if the point at infinity has to be taken away from $\Omega$ because of the behaviour of the hyperbolic radius there.

There are two motivations for the research presented in this paper. The first one is the possibility to understand famous classical results in terms of the gradient of $R$. The classical result of Löwner [18] (compare [26], p. 8/9, also) on convex univalent functions may serve as an example.

THEOREM A: *A domain $\Omega \subset \mathbb{C}$ is convex if and only if*

$$|\nabla R(w,\Omega)| \le 2, \quad w \in \Omega.$$

The second motivation is given by a number of theorems and their applications on the geometry of the surface

$$S_\Omega = \{(w,h) \mid w \in \Omega, h = R(w,\Omega)\}$$

(see [9], [13], [14], [15], [16], [20], [29], [30]). As an example we mention

THEOREM B: *A domain $\Omega \subset \mathbb{C}$ is convex if and only if $R(\cdot,\Omega)$ is a concave function on $\Omega$.*

In a recent paper Kovalev [16] studied an analog of Theorem B for simply connected domains $\Omega \subset \overline{\mathbb{C}}$. He proved that $\mathbb{C} \setminus \Omega$ is a convex set if and only if the function $R(\cdot,\Omega)$ is a locally convex function.

Since the Jacobian of the gradient of $R$ is

$$J(w,\Omega) = \frac{\partial^2 R(w,\Omega)}{\partial u^2}\frac{\partial^2 R(w,\Omega)}{\partial v^2} - \left(\frac{\partial^2 R(w,\Omega)}{\partial u \partial v}\right)^2, \quad w = u + iv \in \Omega,$$

and a condition necessary for a real-analytic function to be convex or concave on $\Omega$ is the inequality

$$J(w,\Omega) \geq 0, \quad w \in \Omega,$$

we observe that Theorem B and its generalizations are related to $\nabla R$. Therefore the present paper is dedicated to the study of the mapping of $\Omega$ by the complex-valued function $\nabla R(\cdot,\Omega)$. To do this, we frequently use the Wirtinger calculus. This, for example, leads to the formula

$$\nabla R(w,\Omega) = \frac{\partial R(w,\Omega)}{\partial u} + i\frac{\partial R(w,\Omega)}{\partial v} = 2\frac{\partial R(w,\Omega)}{\partial \overline{w}}, \quad w = u + iv \in \Omega.$$

The paper is organized as follows. In Section 2 we introduce as a central tool of our analysis the function $\varphi$ meromorphic in $D$ and defined by the equation

$$\frac{f''(z)}{f'(z)} = \frac{2}{\varphi(z) - z}, \quad z \in D \tag{1}$$

(compare [4] and [5]). We prove that $J(f(z), f(D)) \geq 0$ for $z \in D$ if and only if $\varphi$ or $1/\varphi$ is a holomorphic self-map of $D$ (except in the cases where $f(D)$ is a disc or a half-plane), i.e. $\varphi$: $D \to D$, resp. $1/\varphi$: $D \to D$. This enables us to use in what follows the theory of such maps connected with the names of Carathéodory, Denjoy, Julia and Wolff, especially the Grand Iteration Theorem (see [27], p. 78).

In Section 3 we give a unified proof of Theorem B and its generalizations. Moreover, we prove that the gradient map of $\Omega$ is a diffeomorphism if $\Omega$ or $\mathbb{C} \setminus \Omega$ is a convex set, except in the cases where $\Omega$ is a half-plane, a strip or an angular domain.

The fourth section is dedicated to the computation of the sets $G(\Omega) = \nabla R(\Omega,\Omega)$ for polygonal domains $\Omega$. It turns out that in these cases the boundary of $G(\Omega)$ consists of hypocycloids and epicycloids. To avoid confusion we mention that these epicycloids differ from those occurring in the Ptolemaic system.

Section 5 extends the results of Section 3 to the case of doubly connected domains $\Omega$ such that $\mathbb{C}\setminus\Omega$ is a convex set and the point at infinity is an isolated boundary point of $\Omega$. Further, we consider in some detail generalizations to Riemann surfaces.

Section 6 deals with coefficient estimates for functions $f$ that are solutions of (1) in a neighbourhood of the origin for a unimodular bounded holomorphic function $\varphi\colon D \to \overline{D}$ such that $\varphi(0) \neq 0$. In particular, we prove generalizations of estimates proved in [4] and [5].

## 2. An analytic interpretation of $J(w,\Omega) \geq 0$

Let $\varphi$ be the function defined by (1). It is clear that $\varphi$ is not changed by a linear transformation of $f$. Further, a linear transformation of $\Omega$ doesn't influence the behaviour of the hyperbolic radius and its gradient. Actually, if $a\Omega + b = \{aw + b \mid w \in \Omega\}, a \neq 0$, then the above definitions of $R, \nabla R$, and $J$ imply

$$\begin{aligned} R(aw+b, a\Omega+b) &= |a|R(w,\Omega), \\ \nabla R(aw+b, a\Omega+b) &= \exp(i\arg a)\nabla R(w,\Omega), \\ J(aw+b, a\Omega+b) &= |a|^{-2}J(w,\Omega) \end{aligned} \tag{2}$$

and

$$J(w,\Omega) = 4\Big(\Big|\frac{\partial^2 R(w,\Omega)}{\partial\overline{w}\partial w}\Big|^2 - \Big|\frac{\partial^2 R(w,\Omega)}{\partial\overline{w}^2}\Big|^2\Big). \tag{3}$$

The relationship of the function $\varphi$ defined in (1) and the Jacobian $J$ in the form (3) is given by

Lemma 1: *Let $\Omega$ and $f$ be as in the introduction. Then for any $w \in \Omega \setminus \{\infty\}$, $w = f(z)$, $z \in D$, the representation*

$$J(w,\Omega) = 4\frac{(1-|\varphi(z)|^2)^2 - (1-|z|^2)^2|\varphi'(z)|^2}{|\varphi(z)-z|^4|f'(z)|^2} \tag{4}$$

*is valid.*

*Proof:* Using an appropriate conformal automorphism of $D$ one immediately gets with the above abbreviations

$$R(w,\Omega) = |f'(z)|(1-|z|^2)$$

(compare, e.g., [6]). This together with the definition (1) of $\varphi$ yields

$$\frac{\partial R(w,\Omega)}{\partial w} = \frac{|f'(z)|}{f'(z)}\Big(\frac{1-|z|^2}{2}\frac{f''(z)}{f'(z)} - \overline{z}\Big) = \frac{|f'(z)|}{f'(z)}\frac{1-\overline{z}\varphi(z)}{\varphi(z)-z}. \tag{5}$$

Taking partial derivatives of (5) leads to

$$\overline{\frac{\partial^2 R(w,\Omega)}{\partial \overline{w}^2}} = \frac{\partial^2 R(w,\Omega)}{\partial w^2} = \frac{-R(w,\Omega)\varphi'(z)}{(\varphi(z)-z)^2(f'(z))^2}$$

and

$$\frac{\partial^2 R(w,\Omega)}{\partial \overline{w}\partial w} = \frac{1-|\varphi(z)|^2}{|\varphi(z)-z|^2|f'(z)|}.$$

These formulas together with (3) imply (4), the assertion of our Lemma 1. ∎

PROPOSITION 2: *The set of all functions $\varphi$ that are meromorphic in $D$ and satisfy*

$$J_\varphi(z) := (1-|z|^2)|\varphi'(z)| - |1-|\varphi(z)|^2| \le 0 \quad \textit{for any } z \in D$$

*consists of the following three disjoint subsets:*

(a) *There exists $\theta \in [0, 2\pi)$ such that $\varphi(z) = e^{i\theta}, z \in D$,*

(b) *$\varphi$ or $1/\varphi$ is a conformal automorphism of $D$,*

(c) *$\varphi$ or $1/\varphi$ is a holomorphic self-map of $D$ not belonging to (a) or (b).*

*In the cases* (a) *and* (b) *$J_\varphi$ vanishes identically on $D$, in the case* (c) *$J_\varphi$ is negative on $D$.*

*Proof:* We first suppose that there exists $z_0 \in D$ such that $|\varphi(z_0)| = 1$. According to the maximum principle this implies that there exists an arc $\gamma \subset D$ such that $z_0 \in \gamma$ and $|\varphi(z)| = 1$ for all $z \in \gamma$. From the inequality $J_\varphi(z) \le 0$ we deduce that $|\varphi'(z)| = 0$ for all $z \in \gamma$. Hence, $\varphi'(z)=0$ for all $z \in D$ and therefore $\varphi(z) \equiv e^{i\theta}$ for a fixed $\theta \in [0, 2\pi)$. Here, we have obtained case (a) of the proposition.

Suppose now that $|\varphi(z)| \ne 1$ for all $z \in D$. This implies that $\varphi$ or $1/\varphi$ is a holomorphic self-map of $D$ and the inequality $J_\varphi(z) \le 0$ is exactly the Schwarz–Pick inequality for $\varphi$ or $1/\varphi$, respectively. Equality in this inequality is attained in one point if and only if it is attained in all points and $\varphi$ (or $1/\varphi$) is a conformal automorphism of $D$. This leads to case (b).

Now, the rest of the assertion is evident. ∎

Proposition 3: *Let $\Omega$ and $f$ be as in the introduction and let $\varphi$ be defined by* (1). *Then*

(a) $|\varphi(z)| \leq 1$ *for all* $z \in D$ *if and only if* $R(\cdot, \Omega)$ *is a convex function,*

(b) $|\varphi(z)| \geq 1$ *for all* $z \in D$ *if and only if* $R(\cdot, \Omega)$ *is a concave function.*

*Proof:* The formula (5) implies that the equivalences

$$|\varphi(z)|\left\{ \begin{matrix} \leq \\ \geq \end{matrix} \right\}1 \text{ for all } z \in D \Leftrightarrow |\nabla R(w,\Omega)|\left\{ \begin{matrix} \geq \\ \leq \end{matrix} \right\}2 \text{ for all } w \in \Omega$$

are valid. These equivalences together with Liouville's equation show that

$$|\varphi(z)|\left\{ \begin{matrix} \leq \\ \geq \end{matrix} \right\}1 \text{ for all } z \in D \Leftrightarrow \Delta R(w,\Omega)\left\{ \begin{matrix} \geq \\ \leq \end{matrix} \right\}0 \text{ for all } w \in \Omega.$$

Since we know from Lemma 1 and Proposition 2 that the Jacobian of $\nabla R$ is nonnegative in both cases, the sign of the Laplacian of $R$ decides whether $R$ is convex or concave, which completes the proof of our assertion. ∎

We conclude this section with a remark. From (1) it follows that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1 + z/\varphi(z)}{1 - z/\varphi(z)}, \quad z \in D.$$

Hence, the inequality $|\varphi(z)| \geq 1, z \in D$, is equivalent to the classical inequality

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq 0, \quad z \in D,$$

which in turn is equivalent to the convexity of $f(D)$ (see [11], [12], [23]).

Geometrically, the case $|\varphi(z)| \leq 1, z \in D$, is more delicate and rich. Here, one has to distinguish four cases at least.

(1) The function $\varphi$ has a fixed point $\omega \in D$ such that $\varphi'(\omega) = \varphi''(\omega) = 0$.

(2) The function $\varphi$ has a fixed point $\omega \in \partial D$ such that $\varphi'(\omega) \in [0, 1/3]$.

(3) The function $\varphi$ has a fixed point $\omega \in \partial D$ such that $\varphi'(\omega) = 1$.

(4) The remaining cases.

The cases (1) and (2) are connected with several papers that deal with univalent functions $f$ such that $\overline{\mathbb{C}} \setminus f(D)$ is convex (see [4], [5], [12], [16], [17], [19], [22]).

The case (3) leads to doubly connected domains $\Omega$ that have the point at infinity as an isolated boundary point and the case (4) is connected to several aspects of multivalent functions.

## 3. Simply connected domains $\Omega$ with $J(\cdot,\Omega) \geq 0$

In the first assertion of this section we deal with angular domains and strips $\Omega$ such that $\Omega = a\Omega_\alpha + b(a \neq 0, \alpha \in [0,2])$, where the angular domains are translates of

$$\Omega_\alpha = \{w \in \mathbb{C} \setminus \{0\} \mid |\arg w| < \alpha\pi/2\}, \quad \alpha \in (0,2],$$

and the strips are translates of

$$\Omega_0 = \{w \in \mathbb{C} \mid |\operatorname{Im} w| < 1\}.$$

Proposition 4 (compare [15] and [16]): *The Jacobian $J(\cdot,\Omega)$ vanishes identically in $\Omega$ if and only if $\Omega$ is an angular domain or a strip.*

*Proof:* We first prove the sufficiency. Without loss of generality we may consider the domains $\Omega_\alpha, \alpha \in [0,2]$.

In the case $\alpha = 0$ it is known that

$$R(w,\Omega_0) = \frac{4}{\pi}\cos\frac{\pi v}{2}, \quad w = u + iv, \ |v| < 1.$$

Hence

$$\nabla R(w,\Omega_0) = -2i\sin\frac{\pi v}{2}, \quad J(w,\Omega_0) \equiv 0, \ w \in \Omega_0.$$

For $\alpha \in (0,2]$, using the classical case

$$R(\zeta,\Omega_1) = 2\xi, \quad \zeta = \xi + i\eta \in \Omega_1,$$

and the conformal invariance of the hyperbolic metric

$$\frac{|d\zeta|}{R(\zeta,\Omega_1)} = \frac{|dw|}{R(w,\Omega_\alpha)}, \quad w = \zeta^\alpha \in \Omega_\alpha,$$

by straightforward computations we obtain that

$$R(w,\Omega_\alpha) = 2\alpha r\cos\frac{\Theta}{\alpha}, \quad J(w,\Omega_\alpha) \equiv 0, \ w \in \Omega_\alpha,$$

and

$$\nabla R(w,\Omega_\alpha) = 2e^{i\Theta}\Big(\alpha\cos\frac{\Theta}{\alpha} - i\sin\frac{\Theta}{\alpha}\Big), \tag{6}$$

where $w = re^{i\Theta} \in \Omega_\alpha$ and $\alpha \in (0,2]$.

Let now $J(w,\Omega) \equiv 0$ in $\Omega$. In virtue of Lemma 1, $J_\varphi(z) \equiv 0$ in $D$. Hence, we have to examine the cases (a), and (b) of Proposition 2.

Concerning case (a), we have to insert $\varphi(z) \equiv e^{i\theta}$ for a fixed $\theta \in [0, 2\pi)$ into (1) to get

$$\frac{f''(z)}{f'(z)} = \frac{2}{e^{i\theta} - z}, \quad z \in D.$$

Integrating, we obtain that there exist constants $A \neq 0$ and $B$ such that

$$f(z) = \frac{A}{e^{i\theta} - z} + B, \quad z \in D.$$

Thus, $\Omega = f(D)$ is a half-plane.

In case (b) of Proposition 2 for the function $\varphi$ there are two possibilities,

$$\varphi_1(z) = e^{i\theta} \frac{z - z_0}{1 - \overline{z_0} z} \quad \text{or} \quad \varphi_2(z) = e^{i\theta} \frac{1 - \overline{z_0} z}{z - z_0},$$

for fixed $\theta \in [0, 2\pi)$, $z_0 \in D$. Let $f_j$ be a corresponding mapping function defined by (1) with $\varphi = \varphi_j$, $j = 1, 2$.

We first prove that $\varphi_j$ can't have a fixed point $\omega$ in $D$. If $\varphi_j(\omega) = \omega$ for some $\omega \in D$ then $f_j$ has a pole at $z = \omega$. Since $f$ is locally univalent, this is a simple pole. Therefore the function

$$\frac{f_j''(z)}{f_j'(z)} + \frac{2}{z - \omega}$$

has a holomorphic continuation at the point $z = \omega$. On the other hand, formula (1) implies that the residuum of this function at the point $z = \omega$ equals

$$\frac{2\varphi_j'(\omega)}{\varphi_j'(\omega) - 1}.$$

Hence $\varphi_j'(\omega) = 0$, which is impossible for the functions under consideration. Further, these functions have at most two fixed points $\omega_1$ and $\omega_2$ such that $|\omega_1 \omega_2| = 1$. Since $\varphi_j$ has no fixed point in $D$, we conclude that $\omega_1 \in \partial D$ and $\omega_2 \in \partial D$. Now, we again use (1) to see that the function $f_j$ is holomorphic in $\overline{D} \setminus \{\omega_1, \omega_2\}$ and that, according to $|\varphi_j(\tau)| = 1$ for $\tau \in \partial D$,

$$\operatorname{Re}\left(1 + \frac{\tau f''(\tau)}{f'(\tau)}\right) = \operatorname{Re} \frac{\varphi(\tau) + \tau}{\varphi(\tau) - \tau} = 0$$

for $\tau \in \partial D \setminus \{\omega_1, \omega_2\}$. Consequently, the boundary of $f_j(D)$ consists of one or two analytic arcs with vanishing curvature. Hence, $f_j(D)$ is a half-plane or a strip or an angular domain. This completes the proof of Proposition 4. ∎

We now consider a refined version of Theorem B.

THEOREM 5: *The following statements are equivalent.*

(i) *The convex domain $\Omega$ is neither a half-plane nor a strip nor an angular domain.*

(ii) *The hyperbolic radius $R(\cdot, \Omega)$ is a strictly concave function.*

(iii) *The function $\nabla R(\cdot, \Omega)$ is a diffeomorphism of $\Omega$ onto a domain $G$ contained in the disc $D_2 = \{\zeta \mid |\zeta| < 2\}$.*

*Proof:* The equivalence (i) $\Leftrightarrow$ (ii) is a well known result due to Kim, Minda and Wright (see [15] and [20]). We see that this equivalence is a consequence of Propositions 2 and 3, which also prove the implication (iii) $\Rightarrow$ (ii). Therefore, we only have to show that (i) implies (iii).

Let $\Omega$ be a convex domain and let $f\colon D \to \Omega$ be a conformal map of $D$ onto $\Omega$. It is clear that it is sufficient to consider the behaviour of the map $g\colon D \to \Omega$ defined by

$$g(z) = \nabla R(f(z), \Omega), \quad z \in D.$$

Firstly, we suppose that $\Omega$ is a strictly convex domain bounded by a closed analytic curve. In other words, we suppose that $f$ is holomorphic in $\overline{D}$, $f'(z) \neq 0$ for $z \in \overline{D}$, and

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \overline{D}.$$

The corresponding function $\varphi$, defined by (1), satisfies the inequality

$$\mathrm{Re}\, \frac{1 + z/\varphi(z)}{1 - z/\varphi(z)} > 0, \quad z \in \overline{D}.$$

Consequently, $|\varphi(z)| > 1$, $z \in \overline{D}$, $\nabla R(x + iy, \Omega)$ is real-analytic in $\overline{D}$, and $J(f(z), \Omega) > 0$, $z \in \overline{D}$. Therefore, the map $g$ is locally diffeomorphic in $\overline{D}$.

Formula (5) implies that

$$g(e^{i\Theta}) = 2i\exp(i\Psi(\Theta)), \quad \Theta \in [0, 2\pi), \tag{7}$$

where

$$\Psi(\Theta) = \frac{\pi}{2} + \Theta + \arg(f'(e^{i\Theta})). \tag{8}$$

Since

$$|g(e^{i\Theta})| = 2, \quad \Theta \in [0, 2\pi),$$

and

$$\Psi'(\Theta) = \mathrm{Re}\left(1 + \frac{e^{i\Theta} f''(e^{i\Theta})}{f'(e^{i\Theta})}\right) > 0, \quad \Theta \in [0, 2\pi),$$

the map

$$g|_{\partial D}\colon \partial D \to \partial D_2$$

is a diffeomorphism. Due to the argument principle (see, e.g., [21]) we obtain that $g$ is a one-to-one map of $\overline{D}$ onto $\overline{D_2}$. Hence

$$\nabla R(\cdot, \Omega) = g \circ f^{-1}$$

is a diffeomorphism of $\overline{\Omega}$ onto $\overline{G} = \overline{D_2}$.

Now, let $\Omega$ be an arbitrary convex domain with the exceptions mentioned in (i). By Propositions 2 and 4 we know that $J(w, \Omega) > 0$, $w \in \Omega$. For a conformal map $f\colon D \to \Omega$ we will consider a sequence $f_n, n \in \mathbb{N}$, defined by

$$f_n(z) = f(r_n z), \quad r_n = \frac{n}{n+1}, \ n \in \mathbb{N}.$$

For any $n \in \mathbb{N}$, the function $f_n$ is holomorphic on $\overline{D}$, $f_n'(z) \neq 0$, $z \in \overline{D}$, and $\Omega_n = f_n(D)$ is a strictly convex domain bounded by a closed analytic curve (see, e.g., [11]). As we have shown before,

$$g_n = \nabla R(f_n(\cdot), f_n(D)))\colon \overline{D} \to \overline{D_2}$$

is a diffeomorphism. From (5) we conclude that

$$\overline{g_n(z)} = \frac{|f'(r_n z)|}{f'(r_n z)}\Big(r_n(1-|z|^2)\frac{f''(r_n z)}{f'(r_n z)} - 2\overline{z}\Big), \quad z \in \overline{D}.$$

Now we use that $f_n$ together with its derivatives converge as $n \to \infty$ uniformly on any compact set contained in $D$ according to Weierstrass' theorem. Hence $g_n \to g$ as $n \to \infty$ uniformly on any compact set contained in $D$.

To complete the proof of Theorem 5 we observe that $g$ is a real-analytic function, that its Jacobian is positive, since

$$J_g(z, D) = J(f(z), \Omega)|f'(z)|^2 > 0, \quad z \in D,$$

and we use the following lemma.

Lemma 6 (compare [3]): *Let $g$ and $g_n, n \in \mathbb{N}$, be complex-valued functions in $D$ such that $G = g(D), g_n(D) = G_n$ and*

(a) *$g_n\colon D \to G_n$ is a homeomorphism for any $n \in \mathbb{N}$,*

(b) *$g\colon D \to G$ is a locally homeomorphic map,*

(c) *for any $r \in (0,1)$, $D_r = \{z \mid |z| < r\}$, the sequence $g_n|_{\partial D_r}, n \in \mathbb{N}$, converges uniformly on $\partial D_r$ to $g|_{\partial D_r}$ as $n \to \infty$.*

*Then $g\colon D \to G$ is a homeomorphism.*

*Proof:* Let $\zeta_0 \in G$ and let $z_0 \in \{z \mid g(z) = \zeta_0\}$. We have to prove that the set $\{z \mid g(z) = \zeta_0\} \setminus \{z_0\}$ is empty.

The condition (b) of Lemma 6 implies that the set $\{z \mid g(z) = \zeta_0\}$ has no accumulation point in $D$. Hence, there is a sequence $r_j \in (|z_0|, 1)$ such that $g(z) \neq \zeta_0$ for $|z| = r_j$ and $r_j \to 1$ as $j \to \infty$. From condition (c) we conclude that there exists a sequence $n(j) \in \mathbb{N}$ such that $n(j) \to \infty$ as $j \to \infty$ and

$$\max_{|z|=r_j} |g(z) - g_{n(j)}(z)| < \min_{|z|=r_j} |g_{n(j)}(z) - \zeta_0|.$$

By condition (a) the set $\{z \mid g_{n(j)}(z) = \zeta_0\}$ contains one point at most. Therefore, applying Rouché's theorem to the functions $g - g_{n(j)}$ and $g_{n(j)} - \zeta_0$ we see that the sets $\{z \mid g(z) = \zeta_0\} \setminus \{z_0\}$ and $D_{r_j}$ are disjoint. Letting $j \to \infty$ completes the proof of Lemma 6 and, in turn, of Theorem 5. ∎

*Remark:* Evidently, if $\Omega$ is a convex domain and the logarithm of $f'$ of the conformal map $f\colon D \to \Omega$ belongs to the little Bloch space, i.e.

$$\overline{\lim_{|z|\to 1}}(1 - |z|^2)\left|\frac{f''(z)}{f'(z)}\right| = 0,$$

then the gradient image is the disc $D_2$. This is not always the case. For example (see Section 4), the gradient image is a proper subset of the disc $D_2$ if $\Omega$ is a convex domain bounded by a polygon.

We now consider the case of simply connected domains $\Omega$ in $\overline{\mathbb{C}}$ such that $\infty \in \Omega$. Note that the equivalences (i)$\Leftrightarrow$ (ii) in Theorem 7 and in Theorem 8 are due to Kovalev (see [16]).

THEOREM 7: *Let $\Omega$ be a simply connected domain in $\overline{\mathbb{C}}$ such that $\infty \in \Omega$. Then the following statements are equivalent.*

(i) *$\mathbb{C} \setminus \Omega$ is a convex set.*

(ii) *$R(\cdot, \Omega)$ is a strictly convex function in $\Omega \setminus \{\infty\}$.*

(iii) *The gradient image $\nabla R(\Omega, \Omega)$ is an unbounded domain $G$ such that $\infty \in G \subset \{\zeta \mid |\zeta| > 2\}$ and $\nabla R(\cdot, \Omega)$ is a diffeomorphism of $\Omega \setminus \{\infty\}$ onto $G \setminus \{\infty\}$.*

THEOREM 8: *Let $\Omega$ be a simply connected domain in $\mathbb{C}$. Then the following statements are equivalent.*

(i) $\infty \in \partial\Omega$ *and* $\mathbb{C} \setminus \Omega$ *is a convex set with the exception of half-planes and angular domains.*

(ii) $R(\cdot, \Omega)$ *is a strictly convex function in* $\Omega$.

(iii) $\nabla R(\cdot, \Omega)$ *is a diffeomorphism of* $\Omega$ *onto a domain* $G$ *contained in the annulus* $A = \{\zeta \mid 2 < |\zeta| < 4\}$.

The proofs of Theorem 7 and Theorem 8 are similar. As a consequence of (iii) we have in both cases that

$$\frac{|\nabla R(w, \Omega)|}{2} = \left|\frac{\partial R(w, \Omega)}{\partial w}\right| > 1, \quad w \in \Omega.$$

Due to formula (5), we obtain that $|\varphi(z)| < 1$, $z \in D$. This together with Propositions 3 and 4 implies (ii).

The implications (ii) $\Rightarrow$ (i) are proven in [16]. A new proof of them which is necessary for us to identify the remaining cases from Section 1 will be given in Section 5, below. Hence, we have to prove that (i)$\Rightarrow$(iii).

Suppose that (i) holds. Without loss of generality we may assume that $0 \in \Omega$ and consider a conformal map $f\colon D \to \mathbb{C}$ such that $f(\omega) = \infty$, $0 < \omega \leq 1$, and $f$ has an expansion of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad |z| < \omega.$$

The following inequalities are known (see [7], [11], [23], [19], [17], [4], [5]):

$$|a_2| \geq 1, \left|(1 - |z|^2)\frac{f''(z)}{f'(z)} - 2\overline{z}\right| \geq 2, \quad \text{if } \omega \in (0, 1),$$

and

$$2 \geq |a_2| \geq 1, 4 \geq \left|(1 - |z|^2)\frac{f''(z)}{f'(z)} - 2\overline{z}\right| \geq 2, \quad \text{if } \omega = 1.$$

Again, formula (5) implies that $|\nabla R(w, \Omega)| \geq 2$, $w \in \Omega$, if $\infty \in \Omega$ and $4 \geq |\nabla R(w, \Omega)| \geq 2$, $w \in \Omega$, if $\infty \in \partial\Omega$. Since $\nabla R(w, \Omega)$ doesn't vanish for any $w \in \Omega$, we may consider a map $g$ defined by

$$g(z) = \nabla R(f(z), \Omega), \quad z \in D,$$

and, following the proof of (i)$\Rightarrow$(iii) in Theorem 5 with respect to $1/g$, easily get that $g$ is a diffeomorphism of $D$ onto $G$ unless $\Omega$ is a half-plane or an angular domain. As above, this is accomplished in two steps.

STEP 1: If $\infty \in \Omega$ and $\partial\Omega$ is a strictly convex analytic curve, then $1/g$ is a function real-analytic in $D$ such that for the Jacobian of $1/g$ the inequality

$$J_{1/g}(z, D) = \frac{J(f(z), \Omega)|f'(z)|^2}{|\nabla R(f(z), \Omega)|^2} > 0, \quad z \in \overline{D},$$

is valid. Since

$$\frac{1}{g(e^{i\Theta})} = \frac{1}{2}\exp(-i\Psi(\Theta)),$$

where $\Psi$ is an increasing function defined by formula (8), we obtain that $1/g$: $D \to D_{1/2}$ is a diffeomorphism.

STEP 2: We construct a sequence of conformal maps $f_n$: $D \to \Omega_n$, $n \in \mathbb{N}$, such that $f_n \to f$ uniformly on compact subsets in $D$ as $n \to \infty$ and $\partial\Omega_n$ satisfy the condition of Step 1. The existence of such a sequence is trivial if $\infty \in \Omega$. For the case $\infty \in \partial\Omega$ this is proved in [16] and [4]. Since the functions $1/g_n$ defined by

$$\frac{1}{g_n(z)} = \frac{1}{\nabla R(f_n(z), \Omega_n)}$$

converge uniformly on compact subsets of $D$ to $1/g$ as $n \to \infty$, we can use Lemma 6 to complete the proof.

The proofs of Theoem 7 and Theorem 8 are complete.

## 4. Gradient images of polygonal domains

We begin with a simple example. Let $w = f(z) = z + 1/z, z \in D$. Since $\mathbb{C}\setminus f(D) = [-2, 2]$ is a convex set, $\nabla R(\cdot, f(D))$ is a diffeomorphism of $\mathbb{C}\setminus[-2, 2]$ by Theorem 7. Therefore, to find the gradient image it is sufficient to find its boundary. From (1) and (5) it follows that $\varphi(z) = z^3$ and

$$g(z) = \nabla R\Big(z + \frac{1}{z}, f(D)\Big) = 2|1 - z^2|\frac{1 - z\overline{z}^3}{z(1 - \overline{z}^2)^2}, \quad z \in D.$$

Hence,

$$\lim_{z \to e^{i\Theta}} g(z) = \begin{cases} -2i & \text{for any } \Theta \in (0, \pi), \\ 2i & \text{for any } \Theta \in (\pi, 2\pi). \end{cases}$$

Moreover, it is clear that $g(z)$ has no limit as $z \to \pm 1$. Straightforward computations show that the set of all limit values of $g(z)$ as $z \to 1$, $z \in D$, is a curve $\gamma_1$ given by the parametric equation

$$\zeta_1(t) = 2e^{it}\Big(2\cos\frac{t}{2} - i\sin\frac{t}{2}\Big), \quad t \in (-\pi, \pi).$$

This is one branch between the two contact points $2i$ and $-2i$ of an epicycloid that is described by a point on a circle of radius 2 rolling on another circle of radius 2 (compare, e.g., [8], p. 144). Since $g(z) = -g(-z)$, the gradient image of $\overline{\mathbb{C}}\setminus[-2,2]$ is the set of all points outside the two branches of the above epicycloid, where the second branch is described by

$$\zeta_2(t) = -\zeta_1(\pi + t), \quad t \in (-\pi, \pi).$$

One may observe that $\gamma_1$ coincides with the set of values of $\nabla R$ for an angular domain with opening angle $2\pi$, which is a branch of an epicycloid (compare (6)). This observation can be extended: If $\Omega$ is a polygonal domain, then the gradient image is bounded by branches of epicycloids or hypocycloids.

PROPOSITION 9: *Let $\Omega$ be a simply connected domain in $\mathbb{C}$ or in $\overline{\mathbb{C}}$. If the boundary of $\Omega$ is a polygon with inner angles $\pi\alpha_k$, corner points $w_k$ and sides $(w_k, w_{k+1}), k = 1, \ldots, n, w_{n+1} = w_1$, then*

(a) *$\nabla R(\cdot, \Omega)$ is a real-analytic function on $\overline{\Omega} \setminus \{w_1, \ldots, w_n, \infty\}$,*

(b) *the gradient image of a side $(w_k, w_{k+1})$ is a point $\zeta_k$ such that $|\zeta_k| = 2$,*

(c) *the set of all limit values of $\nabla R$ as $w \to w_k$ is a curve $\gamma_k$ that joins $\zeta_{k-1}$ with $\zeta_k$ and is defined by a parametric equation*

$$\zeta_k(t) = 2e^{i(c_k + \alpha_k t)}(\alpha_k \cos t - i \sin t), \quad t \in [-\pi/2, \pi/2],$$

*where $c_k$ is a real constant.*

*Proof:* Let $f$ be a conformal map of $D$ onto $\Omega$. According to the Schwarz–Christoffel formula, the logarithm of $f'$ has an analytic continuation onto $\overline{D} \setminus \{a_1, \ldots, a_n\}$, where $a_k = f^{-1}(w_k)$. This together with formulas (5), (7) and (8) yield (a) and (b).

Moreover, the Schwarz–Christoffel formula implies that the function

$$\frac{f''(z)}{f'(z)} + \frac{1 - \alpha_k}{z - a_k}$$

has an analytic continuation at the point $z = a_k$. Hence, the set of all limit values of $\nabla R$ as $w \to w_k = f(a_k)$ is given by the set of the limit values of the function

$$\exp(i(\alpha_k - 1)\arg(z - a_k))\Big((\alpha_k - 1)\frac{1 - |z|^2}{\overline{z} - \overline{a_k}} - 2z\Big), \quad z \in D,$$

as $z \to a_k$. Direct computations yield (c). This completes the proof of Proposition 9. ∎

## 5. Generalizations to Riemann surfaces and to doubly connected domains

So far we have considered functions $f$ and $\varphi$ meromorphic in $D$ such that (1) was satisfied, i.e.

$$\varphi(z) = z + \frac{2f'(z)}{f''(z)}, \quad z \in D,$$

and $f$ was a universal covering function of a domain $\Omega$.

If $|\varphi(z)| > 1, z \in D$, then $f$ can be shown to be a convex univalent function without assuming that it is a covering function. This is a classical result, as we have seen above.

What can be said about $f$ if (1) is fulfilled, $|\varphi(z)| < 1$, $z \in D$, and we do not impose further conditions on $f$? Unfortunately, we have no general existence theorem for the solutions $f$ of (1) with given $\varphi$. Since (1) relates the fixed points of $\varphi$ to the zeros, poles and singularities of $f'$, it is not surprising that there are close relations between our question and the theory of unimodular bounded holomorphic functions due to Carathéodory, Denjoy, Julia and Wolff. We will need the following statements, which sometimes are quoted as the Grand Iteration Theorem (see [27], p. 78 and compare also [24], p. 82 and [10]).

THEOREM C: *Let $\varphi$ be a holomorphic self-map of $D$ that is not a conformal automorphism of $D$ with a fixed point in $D$. Then there is a unique point $\omega \in \overline{D}$ such that the iterates*

$$\varphi^n(z) = (\varphi \circ \cdots \circ \varphi)(z)$$

*converge to $\omega$ as $n \to \infty$ uniformly on any compact subset of $D$ and*

($C_1$) *if $\omega \in D$, then $\varphi$ has no fixed point in $D \setminus \{\omega\}$, and*

$$\varphi(\omega) = \omega, \quad 0 \le |\varphi'(\omega)| < 1,$$

($C_2$) *if $\omega \in \partial D$ (Denjoy–Wolff point of $\varphi$), then $\varphi$ has no fixed point in $D$ and $\varphi$ and $\varphi'$ have angular limits at $\omega$ such that*

$$\varphi(\omega) = \omega, \quad 0 < \varphi'(\omega) \le 1,$$

*where*

$$\varphi'(\omega) = \sup_{z \in D} \frac{\operatorname{Re} \frac{\omega + z}{\omega - z}}{\operatorname{Re} \frac{\omega + \varphi(z)}{\omega - \varphi(z)}}.$$

Suppose now that $f$ is a function meromorphic in $D$ and that $T$ is a conformal automorphism of $D$. Clearly, the Riemann surfaces $f(D)$ and $(f \circ T)(D)$ are identical.

PROPOSITION 10: *Let $\varphi$ be a holomorphic self-map of $D$ and $T$ a conformal automorphism of $D$. If $f$ is a solution of equation (1), then*

$$\frac{\tilde{f}''(z)}{\tilde{f}'(z)} = \frac{2}{\tilde{\varphi}(z) - z}, \quad z \in D, \tag{9}$$

*where*

$$\tilde{f} = f \circ T \quad \text{and} \quad \tilde{\varphi} = T^{-1} \circ \varphi \circ T.$$

*Proof:* Let

$$T(z) = e^{i\alpha} \frac{z - a}{1 - \overline{a}z},$$

where $\alpha \in [0, 2\pi)$ and $a \in D$ are fixed. By straightforward computations, we get

$$\begin{aligned}\frac{\tilde{f}''(z)}{\tilde{f}'(z)} &= \frac{f''(T(z))}{f'(T(z))} T'(z) + \frac{T''(z)}{T'(z)} = \frac{2T'(z)}{T(\tilde{\varphi}(z)) - T(z)} + \frac{T''(z)}{T'(z)} \\ &= \frac{2}{\tilde{\varphi}(z)) - z} \frac{1 - \overline{a}\tilde{\varphi}(z)}{1 - \overline{a}z} + \frac{2\overline{a}}{1 - \overline{a}z} = \frac{2}{\tilde{\varphi}(z) - z}.\end{aligned}$$

This is the assertion. ∎

Firstly, we shall examine the case ($C_1$). Let

$$\Phi(z) = \exp \int_{\omega}^{z} \Big( \frac{2}{\varphi(t) - t} + \frac{s(\omega) + 1}{t - \omega} \Big) dt,$$

where $s(\omega) = (1 + \varphi'(\omega))/(1 - \varphi'(\omega))$. Note that the integrand is holomorphic at the point $t = \omega$.

LEMMA 11: *Let $\varphi$ be a holomorphic self-map of $D$ such that $\varphi(\omega) = \omega$ and $\varphi'(\omega) \in \overline{D}$ for a point $\omega \in D$. Then the following statements are valid.*

(a) *If $|\varphi'(\omega)| < 1$, then equation (1) has a meromorphic solution $f$ if and only if*

$$s(\omega) = m \in \mathbb{N} \setminus \{0\} \quad \text{and} \quad \Phi^{(m)}(\omega) = 0.$$

*For such a function $\varphi$, any solution $f$ of equation (1) has a pole of order $m$ at the point $z = \omega, f$ is holomorphic in $D \setminus \{\omega\}$, and $f'(z) \neq 0$, $z \in D \setminus \{\omega\}$.*

(b) *If $|\varphi'(\omega)| = 1$, then (1) has no meromorphic solution, i.e. there is no single-valued solution of (1) for an elliptic conformal automorphism of $D$.*

*Proof:* (a) It follows from ($C_1$) that for any point in $D \setminus \{\omega\}$ there exists a solution $f'$ of (1) holomorphic in this point and not vanishing there. But in general these local developments do not fit together to a solution holomorphic and single-valued in $D \setminus \{\omega\}$. This depends on the behaviour of $\varphi$ in the point $z = \omega$. Integrating (1) we find

$$f'(z) = C\frac{\Phi(z)}{(z-\omega)^{s(\omega)+1}}, \quad z \in D,\ C = \text{const}. \neq 0.$$

Clearly, $f'$ is single-valued if and only if $s(\omega) = m \in \mathbb{Z}$. The condition $|\varphi(\omega)| < 1$ implies $m > 0$. Moreover, $f$ is single-valued if and only if $\Phi^{(m)}(\omega) = 0$.

(b) An elliptic automorphism of $D$ has no fixed point on $\partial D$ and is holomorphic on $\overline{D}$. Hence, $f$ is holomorphic and locally univalent on $\partial D$. Since

$$|\varphi(\tau)| = 1 \quad \text{and} \quad \varphi(\tau) - \tau \neq 0, \quad \tau \in \partial D,$$

we obtain that the curvature of the closed analytic curve $w = f(\tau), \tau \in \partial D$, is zero, which is impossible. ∎

*Remark:* We consider the case $m = 1$ of Lemma 11. Here, $\Phi'(\omega) = -\varphi''(\omega) = 0$. Hence, a meromorphic solution of equation (1) exists and has a simple pole at $\omega$ if and only if $\varphi(\omega) = \omega \in D, \varphi'(\omega) = \varphi''(\omega) = 0$ (compare [4]).

Next, we describe solutions of (1) with $\varphi(\omega) = \omega \in D$ using a set of meromorphic functions with known geometric properties.

Let $m \in \mathbb{N} \setminus \{0\}$ and let $\Sigma^c(m)$ be the set of functions $F$ such that $F$ has a pole of order $m$ in the point at infinity, $F$ is holomorphic and locally univalent in $\mathbb{C} \setminus D$ and

$$\operatorname{Re}\left(1 + \frac{\zeta F''(\zeta)}{F'(\zeta)}\right) > 0, \quad \zeta \in \mathbb{C} \setminus D.$$

THEOREM 12 (see [4] for $m = 1$): *The following statements are equivalent.*

(i) *The function $f$ is a meromorphic solution of equation (1) for a holomorphic self-map $\varphi$ of $D$ such that $\varphi(\omega) = \omega \in D, \varphi'(\omega) \in D$.*

(ii) *For a fixed $\omega \in D$ there is a function $F \in \Sigma^c(m)$, $m \in \mathbb{N} \setminus \{0\}$, such that*

$$f(z) = F\left(\frac{1-\omega z}{z-\omega}\right), \quad z \in D.$$

*Proof:* Let

$$T(z) = \frac{z-\omega}{1-\overline{\omega}z}.$$

We consider the functions $\tilde{f} = f \circ T$, $\tilde{\varphi} = T^{-1} \circ \varphi \circ T$. If (i) holds, then $\tilde{\varphi}(0) = 0$ so that, by the Schwarz lemma, the condition $|\tilde{\varphi}(z)| < 1$, $z \in D$, is equivalent to the inequality

$$\operatorname{Re}\frac{\tilde{\varphi}(z) + z}{\tilde{\varphi}(z) - z} < 0, \quad z \in D.$$

From Proposition 10 and Lemma 11 it follows that (i) is equivalent to the following statement.

(iii) The function $\tilde{f}$ is holomorphic and locally univalent in $D \setminus \{0\}$, $\tilde{f}$ has a pole of order $m = (1 + \tilde{\varphi}'(0))/(1 - \tilde{\varphi}'(0))$ at $z = 0$ and

$$\operatorname{Re}\left(1 + \frac{z\tilde{f}''(z)}{\tilde{f}'(z)}\right) < 0, \quad z \in D.$$

Taking $F(\zeta) = \tilde{f}(1/\zeta)$ shows that (ii)$\Leftrightarrow$(iii). This completes the proof of Theorem 12. ∎

*Remark:* It is evident that there is an analog of Theorem 12 in the case that $s(\omega) = m \in \mathbb{N} \setminus \{0\}$, but $\Phi^{(m)}(\omega) \neq 0$. For such a function $\varphi$ the equation (1) has solutions with a logarithmic singularity. For example, if $\varphi(z) = cz^2/(2 - cz)$, $z \in D$, $c \in D \setminus \{0\}$ fixed, then a solution $f$ of (1) is

$$f(z) = 1/z + c \log z.$$

Now, suppose that $\varphi$ has no fixed point in $D$. This implies that (1) has solutions that are holomorphic and locally univalent in $D$. We will look for geometric criteria to describe the Riemann surface $f(D)$. An attractive way is to get $f$ using limits of functions $F \in \Sigma^c(m)$. To do so, we have to approximate functions $\varphi$ satisfying ($C_2$) by functions $\varphi$ satisfying ($C_1$). In principle, one may obtain the following proposition.

PROPOSITION 13: *Let $\varphi$ be a holomorphic self-map of $D$ satisfying ($C_2$), i.e. $\varphi(\omega) = \omega \in \partial D$, $\varphi'(\omega) \in (0, 1]$. Then, the following statements are equivalent.*

(i) $\varphi'(\omega) \in (0, \frac{1}{3}]$.

(ii) *There exists a sequence $\varphi_n$, $n \in \mathbb{N}$, of holomorphic self-maps of $D$ such that $\varphi_n(\omega) = \omega \in D$, $\varphi_n'(\omega) = \varphi_n''(\omega) = 0$ and $\varphi_n \to \varphi$ as $n \to \infty$ uniformly on compact subsets of $D$.*

We have no direct proof of Proposition 13. Via known results it is equivalent to the next theorem.

THEOREM 14: *Let $f$ and $\varphi$ be meromorphic functions satifying equation* (1). *The following statements are equivalent.*

(i) *$f$ is holomorphic and univalent in $D$. $\mathbb{C} \setminus f(D)$ is a convex set. $f$ is continuous on $\overline{D}$ with the exception of a point $\omega \in \partial D$ and $f(z) \to \infty$ as $z \to \omega$.*

(ii) *$\varphi$ is a holomorphic self-map of $D$ that has the Denjoy–Wolff point at the point $\omega = \varphi(\omega) \in \partial D$ such that $\varphi'(\omega) \in (0, \frac{1}{3}]$.*

(iii) *f is a function holomorphic and locally univalent in D such that*

$$\operatorname{Re}\Big(\frac{3}{2}\frac{\omega+z}{\omega-z}-\Big(1+\frac{zf''(z)}{f'(z)}\Big)\Big)>0,\quad z\in D,$$

*for a point* $\omega\in\partial D$.

*Proof:* (The major part of the proof of this theorem is based on an unpublished manuscript of Ch. Pommerenke that he had sent to the authors during discussions on [5].)

The proof of (i)⇒(ii) is given in [4].

For the proof of (ii)⇒(iii) let us denote

$$b=\frac{1}{\varphi'(\omega)}\in[3,\infty)\quad\text{and}\quad q=\frac{f''(z)}{f'(z)}.$$

According to the Julia–Wolff Lemma (see ($C_2$) of Theorem C) the inequality

$$\operatorname{Re}\Big(\frac{1+\varphi(z)}{1-\varphi(z)}-b\frac{1+z}{1-z}\Big)\geq 0,\quad z\in D,$$

is valid. Since

$$\operatorname{Re}\Big(\frac{1+z}{1-z}\Big)=\frac{1-|z|^2}{|1-z|^2}$$

and

$$\operatorname{Re}\Big(\frac{1+\varphi(z)}{1-\varphi(z)}\Big)=\operatorname{Re}\frac{(1+z)q+2}{(1-z)q-2}=\frac{(1-|z|^2)|q|^2-4\operatorname{Re}(1+zq)}{|(1-z)q-2|^2},$$

the last inequality is equivalent to the inequality

$$(1-|z|^2)|q|^2-4\operatorname{Re}(1+zq)\geq b\frac{1-|z|^2}{|1-z|^2}(|1-z|^2|q|^2-4\operatorname{Re}((1-z)q)+4).$$

This yields

$$\begin{aligned}0&\geq(b-1)|q|^2-4\operatorname{Re}\frac{q}{1-\bar z}+\frac{4b}{1-|z|^2}+\frac{4\operatorname{Re}(1+zq)}{1-|z|^2}\\&=(b-1)\Big|q-\frac{2b}{b-1}\frac{1}{1-z}\Big|^2-\frac{4b}{b-1}\frac{1}{|1-z|^2}+\frac{4\operatorname{Re}(1+zq)}{1-|z|^2}.\end{aligned}$$

As $b>1$, we see that the inequality

$$\frac{b}{b-1}\frac{1-|z|^2}{|1-z|^2}-\operatorname{Re}(1+zq)\geq 0,\quad z\in D,\tag{10}$$

is valid, which yields (iii) for $b \geq 3$.

In the proof of (iii)⇒(i) we use that, according to the Riesz–Herglotz formula, (iii) implies the representation

$$\frac{3}{2}\frac{1+z}{1-z} - \left(1 + \frac{zf''(z)}{f'(z)}\right) = \frac{1}{2}\int_{-\pi}^{\pi} \frac{1+e^{-i\Theta}z}{1-e^{-i\Theta}z}d\mu(\Theta),$$

where $\mu$ is a probability measure on $[-\pi, \pi]$ such that $\int_{-\pi}^{\pi} d\mu(\Theta) = 1$. Integration of this formula yields

$$f'(z) = f'(0)(1-z)^{-3}\exp\left(\int_{-\pi}^{\pi} \log(1-e^{-i\Theta}z)d\mu(\Theta)\right).$$

We can approximate $\mu$ by a sequence of point measures $\mu_n, n \in \mathbb{N}$, such that

$$d\mu_n(\Theta_{n_k}) = \alpha_{n_k} \in (0,1]$$

and

$$\sum_{k=1}^{n} \alpha_{n_k} = 1, \Theta_{n_1} < \cdots < \Theta_{n_n} < \Theta_{n_1} + 2\pi.$$

Thus, we get a sequence of holomorphic functions $f_n$ with the derivatives

$$f_n'(z) = f'(0)(1-z)^{-3}\prod_{k=1}^{n}(1-e^{-i\Theta_{n_k}}z)^{\alpha_{n_k}}, \quad z \in D.$$

A little analysis shows that the $f_n$ belong to the family of close-to-convex functions introduced by Kaplan (see, e.g., [23], p. 51). This implies that these functions are univalent in $D$. At the same time, the $f_n$ are conformal maps of $D$ onto a polygonal domain by the Schwarz–Christoffel formula. Since $\alpha_{n_k} \in (0, 1]$, the sets $\mathbb{C} \setminus f_n(D)$ are convex polygonal sets. Letting $n \to \infty$, we get (i). The proof of Theorem 14 is complete. ∎

Next, we shall consider the case $\varphi'(\omega) \in (\frac{1}{3}, 1]$ at the Denjoy–Wolff point of $\varphi$. The solutions of (1) corresponding to such a function $\varphi$ are non-univalent in $D$. Actually, if $f$ is univalent then $\mathbb{C}\setminus f(D)$ is a convex set by Kovalev's theorem [16], so that $\varphi'(\omega) \leq 1/3$ by Theorem 14. The next two assertions show that one has to distinguish between the case $\varphi'(\omega) \in (\frac{1}{3}, 1)$ and the case $\varphi'(\omega) = 1$.

PROPOSITION 15: *Let $\varphi$ be a holomorphic self-map of $D$ that has the Denjoy–Wolff point $\omega = \varphi(\omega) \in \partial D$ such that $\varphi'(\omega) \in (\frac{1}{3}, 1)$. Then any solution $f$ of equation (1) is holomorphic and locally univalent in $D$ and has the following properties.*

(a) *Let* $b = 1/\varphi'(\omega)$ *and*

$$P(z) = \frac{b}{b-1}\frac{\omega+z}{\omega-z} - \Big(1 + \frac{zf''(z)}{f'(z)}\Big), \quad z \in D.$$

*Then* $\operatorname{Re} P(z) > 0$, $z \in D$.

(b) $f$ *is at most* $p$*-valent in* $D$ *with*

$$p \leq \frac{1+\varphi'(\omega)}{1-\varphi'(\omega)},$$

*i.e. for any* $w_0 \in \mathbb{C}$ *the set* $\{z \in D \mid w_0 = f(z)\}$ *contains at most* $p$ *points.*

*Proof:* We get assertion (a) following the proof of inequality (10) in the proof of Theorem 14. The inequality is strict according to the minimum principle for harmonic functions, since $\operatorname{Re}(P(0)) = (b-1)^{-1} > 0$.

For the proof of (b), let $w_0 \in \mathbb{C}$ and $N(w_0, f, r)$ be the number of $w_0$-points in the disc $D_r$. If $f(z) \neq w_0$ for $|z| = r$, $r \in (0,1)$, then

$$N(w_0, f, r) = \frac{1}{2\pi}\int_0^{2\pi} \frac{re^{i\Theta} f'(re^{i\Theta})}{f(re^{i\Theta}) - w_0} d\Theta$$

by the argument principle. Now, we recognize that (1) implies that for any $r \in (0,1)$ the inequality

$$\begin{aligned} b(r) &= \int_0^{2\pi} \Big|\operatorname{Re}\Big(1 + \frac{re^{i\Theta} f''(re^{i\Theta})}{f'(re^{i\Theta})}\Big)\Big| d\Theta \\ &\leq \operatorname{Re}\int_0^{2\pi} \Big(P(re^{i\Theta}) + \frac{b}{b-1}\frac{1+\overline{\omega} re^{i\Theta}}{1-\overline{\omega} re^{i\Theta}}\Big) d\Theta = 2\pi\Big(\frac{2b}{b-1} - 1\Big) \end{aligned}$$

is valid. From Radon's inequality ([25], see [3] for an adaption and [28] for another proof)

$$N(w_0, f, r) < b(r)/2\pi.$$

The last inequalities imply that

$$N(w_0, f, r) < \frac{b+1}{b-1} = \frac{1+\varphi'(\omega)}{1-\varphi'(\omega)}.$$

Letting $r \to 1$ gives the assertion (b). This completes the proof of Proposition 15. ∎

*Remark:* It is known that any covering map $f\colon D \to \Omega$ for a multi-connected domain $\Omega$ is an $\infty$-valent function. Hence, $f$ can be a covering map and a solution of (1) if and only if $\varphi'(\omega) = 1$ at the Denjoy–Wolff point of $\varphi$.

THEOREM 16: *Let $\Omega$ be a doubly connected domain in $\overline{\mathbb{C}}$ and $f\colon D \to \Omega$ an universal covering map of $D$ onto $\Omega$. The following statements are equivalent.*

(i) *The point at infinity is an isolated boundary point of $\Omega$ and $\mathbb{C} \setminus \Omega$ is a convex set.*

(ii) *The function $\varphi$ defined by*

$$\varphi(z) = z + \frac{2f'(z)}{f''(z)}, \quad z \in D,$$

*is a holomorphic self-map of $D$ that has a Denjoy–Wolff point $\omega$ with $\varphi'(\omega) = 1$, and $\varphi$ is not an automorphism of $D$.*

(iii) *$R(\cdot, \Omega)$ is a strictly convex function.*

(iv) *$\nabla R(\cdot, \Omega)$ is a diffeomorphism of $\Omega$ onto a domain $G$ contained in $\{\zeta \mid 2 < |\zeta| < \infty\}$.*

*Proof:* We first prove the chain (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(i) of implications. Suppose (i) holds. The function $f_1$ defined by

$$\zeta = f_1(z) = \exp\frac{1+z}{1-z}, \quad z \in D,$$

is a universal covering map of $D$ onto $E = \{\zeta \mid 1 < |\zeta| < \infty\}$. Because of the convexity of $\mathbb{C} \setminus \Omega$ there exists a function $F \in \Sigma^c(1)$ such that $f = F \circ f_1$. Straightforward computations give

$$R(\zeta, E) = 2|\zeta| \log(|\zeta|), \quad R(w, \Omega) = R(\zeta, E)|F'(\zeta)|,$$

and

$$\frac{\partial R(w,\Omega)}{\partial w} = \frac{|F'(\zeta)|}{F'(\zeta)} e^{-i\Theta}\Big(1 + (1 + \frac{\zeta F''(\zeta)}{F'(\zeta)}) \log(|\zeta|)\Big), \tag{11}$$

where $w = F(\zeta), \zeta = |\zeta|e^{i\Theta} \in E$. Since

$$\operatorname{Re}\left(1 + \frac{\zeta F''(\zeta)}{F'(\zeta)}\right) > 0, \quad \zeta \in E,$$

we get

$$\left|\frac{\partial R(w,\Omega)}{\partial w}\right| \geq 1 + \operatorname{Re}\left(1 + \frac{\zeta F''(\zeta)}{F'(\zeta)}\right) \log(|\zeta|) > 1,$$

so that $|\varphi(z)| \leq 1$, $z \in D$, according to formula (5). Since $\Omega$ is a doubly connected domain and $\infty \notin \Omega$, we conclude that $\varphi$ is a holomorphic self-map of $D$ and there is a Denjoy–Wolff point $\omega$ such that $\varphi'(\omega) = 1$ (see Propositions 4, 12, 15, and Theorem 14) and $J_\varphi(z) < 0, z \in D$. Thus, (ii) holds.

The implication (ii)⇒(iii) is a simple consequence of Proposition 2 and Proposition 3.

The implication (iii)⇒(i) is proved in [16].

Now we prove the chain (i)⇒(iv)⇒(iii) of implications. We see that the implication (iv)⇒(iii) is trivial. Now we show that (i), (ii), and (iii) imply (iv). The function $\nabla R(\cdot,\Omega)$ is locally diffeomorphic in $\Omega$, since $J(w,\Omega) > 0, w \in \Omega$. Next, we follow the proof for simply connected domains with little but important differences. Namely, instead of $\nabla R(f(z),\Omega)$ ($f$ is not univalent) we consider a function $g$ defined by

$$g(\zeta) = \nabla R(F(\zeta),\Omega), \quad \zeta \in E,$$

with a univalent function $F \in \Sigma^c(1)$. Due to (11) the function $g$ is given by an explicit formula. In particular, we have that $g(\zeta) \to \infty$ as $\zeta \to \infty$. Taking

$$\Omega_n = \Big\{F\Big(\Big(1+\frac{1}{n}\Big)\zeta\Big) \mid \zeta \in E\Big\}, \quad n \in \mathbb{N},$$

and

$$g_n(\zeta) = \nabla R\Big(F\Big(\Big(1+\frac{1}{n}\Big)\zeta\Big),\Omega_n\Big), \quad n \in \mathbb{N},$$

we verify that $g_n$ is locally homeomorphic in $\overline{E}\backslash\{\infty\}, g_n(\zeta) \to \infty$ as $\zeta \to \infty$, and $g_n|_{|\zeta|=1}$ is an homeomorphism. Consequently (see, e.g., [2]), $g_n$ is a homeomorphism. Using Lemma 6 applied to $1/g_n(1/z)$ we get that $g$ is a homeomorphism, too. This completes the proof of Theorem 16. ∎

## 6. Estimates for Taylor coefficients

It this section we consider the Taylor coefficients of local developments of solutions of the differential equation (1) for a unimodular bounded function $\varphi$. For the sake of simplicity we restrict our investigations to a neighbourhood of the origin. Concerning this case, in [4] and [5] estimates for the Taylor coefficients of solutions of (1) that are univalent in $D$ have been proved. We generalize these results as follows.

THEOREM 17: *Let $\varphi$ be a holomorphic self-map of $D$ such that $\varphi(0) \neq 0$. Further, let $f$ be a solution of (1) with an expansion*

$$\frac{f(z)-f(0)}{f'(0)} = z + \sum_{k=2}^{\infty} a_k z^k,$$

*which is valid in some neighbourhood of the origin. Then*

$$|a_3 - a_2^2| \leq \frac{|a_2|^2 - 1}{3} \quad \textit{and} \quad |a_n| \geq 1, n \geq 2.$$

*Equality in the first inequality occurs if and only if*

$$f(z) = \frac{\omega}{2\alpha}\left(\left(\frac{1+\overline{\omega}z}{1-\overline{\omega}z}\right)^{\alpha} - 1\right) =: f(\alpha, \omega; z)$$

*for some* $\alpha \geq 1$ *and* $\omega \in \partial D$. $|a_n| = 1$ *for one* $n \geq 2$ *and in turn for all* $n \geq 2$ *occurs if and only if* $f(z) = f(1, \omega; z)$ *for some* $\omega \in \partial D$.

*Proof:* The proof for $|a_n| \geq 1, n \geq 2$, exactly follows the proof for this inequality for functions $f$ meromorphic and univalent in $D$ that was given in [5]. For the proof in the general case we only need a slightly stronger version of the coefficient results on quasi-subordinate function. This is the content of the next theorem that generalizes results of Rogosinski and Robertson and uses the Littlewood theorem on subordinate functions for its proof, as usual (for details see [23], chapter 2 and [5]).

THEOREM 18: *Let the functions* $F$ *and* $G$ *be defined and holomorphic in a neighbourhood of the origin, where they have the expansions*

$$F(z) = \sum_{n=0}^{\infty} A_n z^n \quad \textit{and} \quad G(z) = \sum_{n=0}^{\infty} B_n z^n.$$

*If there exist two unimodular bounded holomorphic functions* $\varphi_1$ *and* $\varphi_2$ *such that*

$$F(z) = \varphi_1(z) G(z\varphi_2(z))$$

*in a neighbourhood of the origin, then for any* $n \geq 0$ *the inequality*

$$\sum_{k=0}^{n} |A_k|^2 \leq \sum_{k=0}^{n} |B_k|^2$$

*is valid.*

For the proof of $|a_3 - a_2^2| \leq (|a_2|^2 - 1)/3$, we use a local version of Lemma 1 and Proposition 2 and the condition that $\varphi$ is a unimodular bounded holomorphic function. Straightforward computations show that

$$J_{\varphi}(0) \leq 0 \tag{12}$$

is equivalent to the inequality in question. Equality at the origin occurs in (12) if and only if the Jacobian vanishes identically. As in the proof of Proposition

4, we obtain that this is true if and only if $f(z) = f(\alpha, \omega; z)$ where $\alpha$ and $\omega$ are as above. This completes the proof of Theorem 17. ∎

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